On Exponential Convergence of Coordination Learning Control for Multi-agent Formation

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Abstract

The exponential convergence problem is studied for coordination learning control of multi-agent formation under the switching network topology. A necessary and sufficient condition on exponential convergence is presented for coordination learning control algorithms of multi-agent formation tasks without any reference as prior knowledge. Moreover, it is shown that the developed results are effective for agents with higher-order dynamics. Numerical simulations are performed to illustrate the coordination learning performance of networked agents.

Keywords: Coordination Learning Control; Multi-agent System; Formation; Exponential Convergence; Switching Network

1 Introduction

Coordination learning control for multi-agent systems has been considered as one of the most compelling research topics due to its theoretical importance and wide applications (see, e.g., [1-15]). Recently, it has attracted much attention to incorporate iterative learning control technique into the coordination control for multi-agent systems [7-15]. Since iterative learning results in fundamentally a two-dimensional system with respect to both time and iteration, an additional iterative process is required to implement the consensus tasks in [12] which also impose the multi-agent networks possessing ceratin repetitive (periodic or cyclic) properties. In addition, it is worth noting that the monotonic convergence is a desirable objective to guarantee good learning transient behaviors for the iterative learning processes since this property guarantees the input signal to be improved better and better with the increasing of iteration. In particular, a new class of iterative-learning-approaches-based coordination learning problems have been studied for multi-agent formation under the switching directed networks (see, e.g., [13, 15]).

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It has been shown that two independent dynamics of the multi-agent coordination learning processes can be clearly revealed by a class of two-dimensional (2-D) Roesser systems which are closely related to the multi-agent networks [13]. Through developing some new convergence results for this class of networked 2-D Roesser systems, the coordination learning results have been proposed for two relative formation issues of agents both with and without a reference trajectory as prior knowledge. However, in [13], a sufficient condition is only provided to guarantee the exponential convergence for the relative formation tasks of agents in the absence of the reference trajectory, in comparison with a necessary and sufficient condition developed in the presence of the reference trajectory. By this observation, we consider the coordination learning problem of the relative formation task of agents in the absence of the reference trajectory, and present a necessary and sufficient condition to achieve its exponential convergence. Moreover, we will consider the agents with general high-order dynamics, rather than the first-order dynamic assumed in [13].

2 Problem Statement

Consider the repetitive systems with $n$ mobile agents which are labeled 1 through $n$. Let every agent share the common state space $\mathbb{R}^m$, where $m \geq 1$ is the order of the agent’s dynamics. In [13], the discussions were presented in the particular case where $m = 1$, but it was also remarked that its established results can be extended to the general case where $m \geq 1$. This paper considers the extension with the help of the Kronecker product. In the following, $I_m$ denotes the $m$th-order identity matrix, $A \otimes B$ denotes the Kronecker product of matrices or vectors $A$ and $B$, and $\|A\|_2$ denotes the Euclidean (respectively, spectral) norm of a vector (respectively, matrix) $A$. If there are no special indications, we use other notations or symbols of [13] for the readers’ convenience.

The $k$-th agent is considered to have the following dynamics over a finite time duration $i \in \mathbb{Z}_N$ and along an infinite iteration process $j \in \mathbb{Z}_+$ [13]:

$$x_k(i + 1, j) = x_k(i, j) + u_k(i, j), \quad k \in \mathbb{I}_n, \quad (1)$$

where the state $x_k(i, j)$ has the initial vector value: $x_k(0, j) = x_k(0) \Delta x_{k0}$, and the control input $u_k(i, j)$ is to be designed. Let every agent be regarded as a vertex in the $n$-th-order directed graph $G(i) = (\mathcal{V}, \mathcal{E}(i), \mathcal{A}(i))$ switching with respect to the time step $i$ on $\mathbb{Z}_N$, where $\mathcal{V} = \{v_k : k \in \mathbb{I}_n\}$ is the vertex set, $\mathcal{E}(i) \subseteq \{(v_k, v_l) : v_k, v_l \in \mathcal{V}\}$ is the edge set, and $\mathcal{A}(i) = [a_{kl}(i)] \in \mathbb{R}^{n \times n}$ is the nonnegative weighted adjacency matrix. If $(v_k, v_l) \in \mathcal{E}(i)$, then at the time step $i$, there exists an available information channel from $v_l$ to $v_k$, and $v_l$ is a neighbor of $v_k$. The index set of all the neighbors of $v_k$ at the time step $i$ is denoted by $\mathcal{N}_k(i) = \{l : (v_k, v_l) \in \mathcal{E}(i)\}$. Also, $a_{kl}(i) > 0$ if and only if $(v_k, v_l) \in \mathcal{E}(i), a_{kl}(i) = 0$ if and only if $(v_k, v_l) \notin \mathcal{E}(i)$, and $a_{kk}(i) = 0$ for $\forall k, l \in \mathbb{I}_n$ and $i \in \mathbb{Z}_N$. For the directed graph $G(i)$, the Laplacian matrix associated with $\mathcal{A}(i)$ is defined as $L_{\mathcal{A}(i)} = [L_{kl}(i)] \in \mathbb{R}^{n \times n}$, where $L_{kl}(i) = \sum_{l \in \mathcal{N}_k(i)} a_{kl}(i)$ if $l = k$ and $L_{kl}(i) = -a_{kl}(i)$ if $l \neq k$.

For every agent $v_k$, let $\tau_k(i) \in \mathbb{R}^m$ represent the desired deviation trajectory of it over $i \in \mathbb{Z}_N$. We employ the difference $\tau_k(i) = \tau_k(i) - \tau(i)$ to represent the desired relative formation between the agent $v_k$ and the agent $v_l$. Moreover, we use $x_{kl}(i, j) = x_k(i, j) - x_l(i, j)$ to denote the relative formation between the agent $v_k$ and the agent $v_l$ at the iteration $j$. Then the coordination learning problem addressed in this paper is to find an appropriate algorithm such that the relative formation between any two agents can follow the corresponding desired formation, i.e., for $\forall k, l \in \mathbb{I}_n$,

$$\lim_{j \to \infty} x_{kl}(i, j) = \tau_{kl}(i), \quad i = 1, 2, \ldots, N \quad (2)$$
and the limit of the formation objective (2) can be approached exponentially fast with the increase of iteration number \( j \) (i.e., \( \| x_{kl}(i) - x_{kl}(i, j) \|_2 \leq b_x \gamma^j \) for some bounded \( b_x \geq 0 \) and \( \gamma \in [0, 1) \), where \( i = 1, 2, \ldots, N \)).

For the formation objective (2), we apply the following algorithm given using iterative rules:

\[
u_k(i, j + 1) = u_k(i, j) + \sum_{l \in N_k(i)} \phi_{kl}(i) a_{kl}(i) [\tau_{kl}(i + 1) - x_{kl}(i + 1, j)]\quad k \in \mathcal{I}_n, \tag{3}
\]

where \( u_k(i, 0) \) is a bounded initial input that can be arbitrarily prescribed, and \( \phi_{kl}(i) \) is a scalar gain with the following design criteria [13]:

\[
\phi_{kl}(i) \begin{cases} > 0, & l \in N_k(i) \\ = 0, & \text{otherwise} \end{cases}, \quad k \in \mathcal{I}_n.
\]

Note that the algorithm (3) uses the information \( x_{kl}(i + 1, j) \) at the time step \( i + 1 \) to update the control input at the time step \( i \). However, as pointed out in [13], the information \( x_{kl}(i + 1, j) \) is from the \( j \)-th iteration, and therefore is available at the iteration \( j + 1 \) when designing \( u_k(i, j + 1) \).

This actually implies the causality of the algorithm (3) by taking advantage of the idea of iterative learning control (ILC) with anticipation in time. For convenience, we use \( \Phi(i) = [\phi_{kl}(i)] \in \mathbb{R}^{n \times n} \) to denote a unified matrix consisting of the scalar gains of the algorithm (3).

### 3 Main Results

#### 3.1 Preliminary results

Let \( \eta_k(i, j) = x_k(i, j + 1) - x_k(i, j) \) and \( \zeta_k(i, j) = \tau_k(i) - x_k(i, j) \). Then by applying (3) to (1), we can follow the similar steps used in [13] to arrive at

\[
\eta_k(i + 1, j) = \eta_k(i, j) + \sum_{l \in N_k(i)} \phi_{kl}(i) a_{kl}(i) [\zeta_k(i + 1, j) - \zeta_l(i + 1, j)]
\tag{4}
\]

\[
\zeta_k(i + 1, j + 1) = -\eta_k(i, j) + \zeta_k(i + 1, j) - \sum_{l \in N_k(i)} \phi_{kl}(i) a_{kl}(i) [\zeta_k(i + 1, j) - \zeta_l(i + 1, j)]. \tag{5}
\]

If \( \eta(i, j) = [\eta_1^T(i, j), \eta_2^T(i, j), \ldots, \eta_n^T(i, j)]^T \) and \( \zeta(i, j) = [\zeta_1^T(i, j), \zeta_2^T(i, j), \ldots, \zeta_n^T(i, j)]^T \) are denoted, then the facts of (4) and (5) can be reformulated in a compact 2-D Roesser system of

\[
\begin{bmatrix}
\eta(i + 1, j) \\
\zeta(i + 1, j + 1)
\end{bmatrix} =
\begin{bmatrix}
I_{nm} & \mathcal{L}_{\Phi(i) \circ \mathcal{A}(i)} \otimes I_m \\
-I_{nm} & I_{nm} - \mathcal{L}_{\Phi(i) \circ \mathcal{A}(i)} \otimes I_m
\end{bmatrix}
\begin{bmatrix}
\eta(i, j) \\
\zeta(i + 1, j)
\end{bmatrix} \tag{6}
\]

Where \( \mathcal{L}_{\mathcal{A}(i) \circ \Phi(i)} \) denotes the Laplacian matrix associated with \( \mathcal{A}(i) \circ \Phi(i) \), \( \eta(0, j) = 0 \) holds for \( j \in \mathbb{Z}_+ \), and \( \zeta(i + 1, 0) \) is bounded for \( i \in \mathbb{Z}_{N-1} \) (i.e., there exits a constant bound \( b_\zeta \geq 0 \) such that \( \max_{i \in \mathbb{Z}_{N-1}} \| \zeta(i + 1, 0) \|_2 \leq b_\zeta \)). Based on the development of the 2-D Roesser system (6), we can propose the following lemma to give some preliminary results.
Lemma 1 Let $G(i) \in \mathbb{R}^{n \times (n-1)}$ be any matrix to form a nonsingular matrix $P(i) = [1_n \ G(i)] \in \mathbb{R}^{n \times n}$, and then $E(i) \in \mathbb{R}^{1 \times n}$ and $F(i) \in \mathbb{R}^{(n-1) \times n}$ be the two matrices to form the inverse matrix of $P(i)$ as $P^{-1}(i) = [E^T(i) \ F^T(i)]^T \in \mathbb{R}^{n \times n}$. In particular, we can use $E(i)$, $F(i)$ and $G(i)$ as

$$
E(i) \equiv [1 \ 0 \ \cdots \ 0] \in \mathbb{R}^{1 \times n} \\
F(i) \equiv [-1_{n-1} \ I_{n-1}] \in \mathbb{R}^{(n-1) \times n} \\
G(i) \equiv \begin{bmatrix} 0 \\ I_{n-1} \end{bmatrix} \in \mathbb{R}^{n \times (n-1)}
$$

for $\forall i$. (7)

If we denote $e(i, j) = [F(i) \otimes I_m] \zeta(i + 1, j)$ for $i \in \mathbb{Z}_{N-1}$ and $j \in \mathbb{Z}_+$, then

(a) The formation objective (2) is achieved (exponentially fast) if and only if $\lim_{j \to \infty} e(i, j) = 0$ for $i \in \mathbb{Z}_{N-1}$ is guaranteed (exponentially fast);

(b) The following 2-D Roesser system with respect to $\eta(i, j)$ and $e(i, j)$ can be developed:

$$
\begin{bmatrix}
\eta(i + 1, j) \\
e(i, j + 1)
\end{bmatrix} =
\begin{bmatrix}
I_{nm} & \mathcal{L}_{\Phi(i)\mathcal{A}(i)} G(i) \otimes I_m \\
-F(i) \otimes I_m & I_{(n-1)m} - F(i) \mathcal{L}_{\Phi(i)\mathcal{A}(i)} G(i) \otimes I_m
\end{bmatrix}
\begin{bmatrix}
\eta(i, j) \\
e(i, j)
\end{bmatrix}
$$

(8)

Where $\eta(0, j) = 0$ for $j \in \mathbb{Z}_+$ and $e(i, 0)$ is bounded for $i \in \mathbb{Z}_{N-1}$ (i.e., we have a constant bound $b_0 = b_\zeta \max_{i \in \mathbb{Z}_{N-1}} \|F(i) \otimes I_m\|_2$ such that $\max_{i \in \mathbb{Z}_{N-1}} \|e(i, 0)\|_2 \leq b_0$).

Proof This lemma can be summarized by noting the discussion of Section 3 and the proof of Theorem 9 in [13], and thus its proof is omitted here.

Note that the 2-D Roesser system (8) can be seen as a reduced-order form of the 2-D Roesser system (6) which describes the two independent dynamics of the coordination learning system (1) and (3). With the result (a) of Lemma 1, we know that the considered formation objective (2) can be equivalently transformed into the vertical stability (see [13]) problem of the reduced-order 2-D Roesser system (8).

3.2 Convergence results

For the multi-agent system (1) under the formation algorithm (3), we can propose the following convergence result for the coordination learning problem (2).

Theorem 1 Consider the multi-agent system (1), and apply the algorithm (3) with

$$
\sum_{l \in \mathcal{N}_k(i)} \phi_{kl}(i) a_{kl}(i) < 1, \quad \forall k \in \mathcal{I}_n.
$$

(9)

Then not only can the formation objective (2) be achieved but also its limit can be approached exponentially fast if and only if the directed graph $\mathcal{G}(i)$ has a spanning tree for $i \in \mathbb{Z}_{N-1}$.

Remark 1 Comparing with Theorem 2 of [13], the above Theorem 1 also makes the exponential convergence of the formation objective (2) be achieved under a necessary and sufficient condition.
This theorem actually shows that the stability condition provided in Theorem 2 of [13] also ensures the exponential convergence of the formation objective (2) when applying the algorithm (3) under the gain selection condition (9). As stated in [13], there always exist nonzero learning gains \( \phi_{kl}(i) \) to fulfill (9) for \( k \in \mathcal{I}_n \), and we can consider an alternative selection of them as

\[
\phi_{kl}(i) = \frac{\theta_k}{|\mathcal{N}_k(i)|} \sigma_{kl}(i),\quad \text{if} \ l \in \mathcal{N}_k(i),
\]

where \(|\mathcal{N}_k(i)|\) is the number of neighbors of the agent \( v_k \) at the time step \( i \), and \( \theta_k \in (0, 1) \).

**Proof of Theorem 1** It can be easily seen that the necessity of this theorem is a consequence of that of Theorem 2 of [13]. Next, we only prove the sufficiency by adopting the inductive analysis approach with respect to \( i \). To this end, we use the 2-D Roesser system (8) to obtain

\[
e(i, j + 1) = [I_{(n-1)m} - F(i)\mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}G(i) \otimes I_m] \ e(i, j) - [F(i) \otimes I_m] \eta(i, j), \tag{10}
\]

where \( \eta(0, j) = 0 \) and \( \eta(i, j) \) for \( i \geq 1 \) satisfies

\[
\eta(i, j) = \sum_{l=0}^{i-1} [\mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}G(l) \otimes I_m] \ e(l, j). \tag{11}
\]

In addition, we can verify that \( I_n - \mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}} \) is a stochastic matrix under the condition (9). Since \( \mathcal{G}(i) \) has a spanning tree, we can further obtain that \( I_n - \mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}} \) has only one eigenvalue equal to one, and all its other eigenvalues have modulus less than one. Using Lemma 1, we have

\[
P^{-1}(i) \left[ I_n - \mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}} \right] P(i) = \begin{bmatrix}
E(i) & [I_n - \mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}] 1_n & E(i) & [I_n - \mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}] G(i) \\
F(i) & [I_n - \mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}] 1_n & F(i) & [I_n - \mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}] G(i) \\
1 & E(i) & [I_n - \mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}] G(i) \\
0 & I_{n-1} - F(i)\mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}G(i) & 
\end{bmatrix},
\]

where \( \mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}1_n = 0 \) is also inserted. These two facts imply \( \rho \left(I_{n-1} - F(i)\mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}G(i)\right) < 1 \), which leads to

\[
\rho \left(I_{(n-1)m} - F(0)\mathcal{L}_{\Phi(0)_{\circ \mathcal{A}(0)}}G(0) \otimes I_m\right) = \rho \left(\left[I_{(n-1)m} - F(i)\mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}G(i)\right] \otimes I_m\right) = \rho \left(I_{n-1} - F(i)\mathcal{L}_{\Phi(i)_{\circ \mathcal{A}(i)}}G(i)\right) < 1, \quad \forall i \in \mathbb{Z}_{N-1}. \tag{12}
\]

With the above developments, we divide the following proof procedure into three steps.

**Step (i)**. For \( i = 0 \), we will show that \( \lim_{j \to \infty} e(0, j) = 0 \) holds with its limit being approached exponentially fast.

In this case, we can derive from the facts (10) and (12) that

\[
e(0, j + 1) = [I_{(n-1)m} - F(0)\mathcal{L}_{\Phi(0)_{\circ \mathcal{A}(0)}}G(0) \otimes I_m] \ e(0, j), \tag{13}
\]

where \( \rho \left(I_{(n-1)m} - F(0)\mathcal{L}_{\Phi(0)_{\circ \mathcal{A}(0)}}G(0) \otimes I_m\right) < 1 \). This obviously means that all the eigenvalues of \( I_{(n-1)m} - F(0)\mathcal{L}_{\Phi(0)_{\circ \mathcal{A}(0)}}G(0) \otimes I_m \) have modulus less than one. Therefore, it follows from the standard Lyapunov stability theory that, for any positive definite matrix \( 0 < Q \in \mathbb{R}^{(n-1)m \times (n-1)m} \), there exists a unique positive definite matrix \( 0 < \Theta \in \mathbb{R}^{(n-1)m \times (n-1)m} \) satisfying the following Lyapunov equation:

\[
[I_{(n-1)m} - F(0)\mathcal{L}_{\Phi(0)_{\circ \mathcal{A}(0)}}G(0) \otimes I_m]^T \Theta \left[I_{(n-1)m} - F(0)\mathcal{L}_{\Phi(0)_{\circ \mathcal{A}(0)}}G(0) \otimes I_m\right] = -Q. \tag{14}
\]
Using the fact (14), we can easily see that $\Theta - Q$ is positive semi-definite. This leads to

$$0 < \lambda_{\text{min}}(Q) \leq \lambda_{\text{min}}(\Theta) \leq \lambda_{\text{max}}(\Theta),$$  \hspace{1cm} (15)

where $\lambda_{\text{min}}(\cdot)$ and $\lambda_{\text{max}}(\cdot)$ denote the minimum and maximum eigenvalues of any certain matrix, respectively. Let us consider the Lyapunov function $V(e(0, j)) = e^T(0, j)\Theta e(0, j)$ for the system (13). By noting the positive definiteness of $Q$ and $\Theta$, we have

$$e^T(0, j)Qe(0, j) \geq \lambda_{\text{min}}(Q)e^T(0, j)e(0, j) = \frac{\lambda_{\text{min}}(Q)}{\lambda_{\text{max}}(\Theta)} \left[ \lambda_{\text{max}}(\Theta)e^T(0, j)e(0, j) \right]$$

$$\geq \frac{\lambda_{\text{min}}(Q)}{\lambda_{\text{max}}(\Theta)}(0, j)e(0, j)$$

$$= \frac{\lambda_{\text{min}}(Q)}{\lambda_{\text{max}}(\Theta)}V(e(0, j)).$$  \hspace{1cm} (16)

By considering $V(e(0, j))$ for the system (13) and then applying the facts of (14) and (16), we can obtain

$$V(e(0, j + 1)) = V(e(0, j)) + e^T(0, j)\Theta e(0, j + 1) - e^T(0, j)\Theta e(0, j)$$

$$= V(e(0, j)) + e^T(0, j)\left[ I_{(n-1)m} - F(0)L_{\Phi(0)\circ A(0)}G(0) \otimes I_m \right]^T$$

$$\times \Theta \left[ I_{(n-1)m} - F(0)L_{\Phi(0)\circ A(0)}G(0) \otimes I_m \right]$$

$$= V(e(0, j)) - e^T(0, j)Qe(0, j)$$

$$\leq [1 - \lambda_{\text{min}}(Q)/\lambda_{\text{max}}(\Theta)]V(e(0, j)),$$

which, together with $\lambda_{\text{min}}(\Theta)\|e(0, j)\|^2 \leq V(e(0, j)) \leq \lambda_{\text{max}}(\Theta)\|e(0, j)\|^2$ and $\|e(0, 0)\| \leq b_{e0}$ (see the result (b) of Lemma 1), can be used to deduce

$$\|e(0, j)\|^2 \leq \sqrt{V(e(0, j))}/\lambda_{\text{min}}(\Theta) \leq \sqrt{V(e(0, 0))}/\lambda_{\text{min}}(\Theta) \left( \sqrt{1 - \lambda_{\text{min}}(Q)/\lambda_{\text{max}}(\Theta)} \right)^j$$

$$\leq b_{e0}\sqrt{\lambda_{\text{max}}(\Theta)/\lambda_{\text{min}}(\Theta) \left( \sqrt{1 - \lambda_{\text{min}}(Q)/\lambda_{\text{max}}(\Theta)} \right)^j}.\hspace{1cm} (17)$$

Since the fact (15) ensures $0 \leq 1 - \lambda_{\text{min}}(Q)/\lambda_{\text{max}}(\Theta) < 1$ (hence, $0 \leq \sqrt{1 - \lambda_{\text{min}}(Q)/\lambda_{\text{max}}(\Theta)} < 1$), we can easily derive from (17) that $\lim_{j \to \infty} e(0, j) = 0$ holds with its limit being approached exponentially fast.

**Step (ii).** For any given $i \in Z_{N-1}$, we first assume that $\lim_{j \to \infty} e(l, j) = 0$ holds with its limit being approached exponentially fast at the time steps $l = 0, 1, \ldots, i - 1$. Then we will prove that $\lim_{j \to \infty} e(i, j) = 0$ holds with its limit being approached exponentially fast.

For the hypothesis made at the time steps $l = 0, 1, \ldots, i - 1$, we denote $\|e(l, j)\| \leq \tilde{b}(l)\tilde{\gamma}(l)$ for some bounded $\tilde{b}(l) \geq 0$ and $\tilde{\gamma}(l) \in [0, 1]$. Then let $\tilde{\gamma}_{\text{max}} = \max_{l=0}^{i-1}\{\tilde{\gamma}(l)\}$, and we can easily see that $\tilde{\gamma}_{\text{max}} \in [0, 1]$. By inserting this into (11), we can obtain

$$\|\eta(i, j)\| \leq \sum_{l=0}^{i-1} \|L_{\Phi(l)\circ A(l)}G(l) \otimes I_m\|\|e(l, j)\| \leq \sum_{l=0}^{i-1} \|L_{\Phi(l)\circ A(l)}G(l) \otimes I_m\| \tilde{b}(l)\tilde{\gamma}(l) \leq \tilde{b}\tilde{\gamma}_{\text{max}},$$  \hspace{1cm} (18)

where $\tilde{b} = \sum_{l=0}^{i-1} \|L_{\Phi(l)\circ A(l)}G(l) \otimes I_m\|\tilde{b}(l)$ is bounded. Since the condition (12) holds, we can consider the 2-D system theory (see, e.g., Lemma 2 of [13]) for the 2-D Roesser system (8) to deduce that $\lim_{j \to \infty} \lambda_{\text{min}}(e(i, j)) = 0$ and $e(i, j)$ is bounded for all $j \in Z_+$. By this fact, we denote $\|e(i, j)\| \leq \tilde{b}_e$ for some bounded $\tilde{b}_e \geq 0$ and for all $j \in Z_+$. Moreover, the condition (12) implies that all the eigenvalues of $I_{(n-1)m} - F(i)L_{\Phi(i)\circ A(i)}G(i) \otimes I_m$ have modulus less than one. This together with the standard Lyapunov stability theory guarantees that, for any given positive definite matrix $0 < \Xi \in \mathbb{R}^{(n-1)m \times (n-1)m}$, there exists a unique positive definite matrix $0 < \Xi \in \mathbb{R}^{(n-1)m \times (n-1)m}$ satisfying the following Lyapunov equation:

$$[I_{(n-1)m} - F(i)L_{\Phi(i)\circ A(i)}G(i) \otimes I_m]^T\Xi[I_{(n-1)m} - F(i)L_{\Phi(i)\circ A(i)}G(i) \otimes I_m] - \Xi = -\Omega.$$  \hspace{1cm} (19)
Then by again considering (22), we can further derive exponentially fast under the condition (9) if result (a) of Lemma 1, guarantees that the formation objective (2) holds with its limit being approached i

where $V(e(i,j)) = e^T(i,j)\Xi e(i,j)$ is the Lyapunov function for the system (10). By considering $V(e(i,j))$ for the system (10) and inserting (18), (19) and (21), we can derive

$$V(e(i,j + 1)) = V(e(i,j)) + [e^T(i,j + 1)\Xi e(i,j + 1) - e^T(i,j)\Xi e(i,j)]$$

$$= V(e(i,j)) + e^T(i,j)\left\{ I_{(n-1)m} - F(i)\mathcal{L}_{\Phi(i)\circ A(i)}G(i) \otimes I_m \right\}^T$$

$$\times \Xi [I_{(n-1)m} - F(i)L_{\Phi(i)\circ A(i)}G(i) \otimes I_m] - \Xi] e(i,j)$$

$$- 2e^T(i,j) [I_{(n-1)m} - F(i)L_{\Phi(i)\circ A(i)}G(i) \otimes I_m]_\Xi [F(i) \otimes I_m] \eta(i,j)$$

$$+ \eta^T(i,j) [F^T(i) \otimes I_m] \Xi [F(i) \otimes I_m] \eta(i,j)$$

$$\leq V(e(i,j)) - e^T(i,j)\Omega e(i,j)$$

$$+ \left\| 2e^T(i,j) [I_{(n-1)m} - F(i)L_{\Phi(i)\circ A(i)}G(i) \otimes I_m]_\Xi [F(i) \otimes I_m] \eta(i,j) \right\|_2$$

$$+ \left\| \eta^T(i,j) [F^T(i) \otimes I_m] \Xi [F(i) \otimes I_m] \eta(i,j) \right\|_2$$

$$\leq [1 - \lambda_{\min}(\Omega)/\lambda_{\max}(\Xi)] V(e(i,j)) + b_\eta \gamma^2_{\max},$$

where $b_\eta \geq 0$ is given by

$$b_\eta = 2 \left\| I_{(n-1)m} - F(i)L_{\Phi(i)\circ A(i)}G(i) \otimes I_m \right\|_2 \||\Xi\|_2 \left\| F(i) \otimes I_m \right\|_2 b_\eta \gamma_{\max} \||\Xi\|_2 \left\| F(i) \otimes I_m \right\|_2 \|b_\eta \gamma^2_{\max}$$

and $\|\eta(i,j)\|_2 \leq \tilde{b}_\eta \gamma_{\max}$ for all $j \in \mathbb{Z}^+$ due to $\gamma_{\max} \in [0, 1]$ is also inserted (see (18)). From (20), we know $0 \leq 1 - \lambda_{\min}(\Omega)/\lambda_{\max}(\Xi) < 1$. We assume $1 - \lambda_{\min}(\Omega)/\lambda_{\max}(\Xi) \neq \gamma_{\max}$ (otherwise, we can choose a new scalar $\tilde{\gamma}_{\max} \in (\tilde{\gamma}_{\max}, 1)$ to replace $\gamma_{\max}$ for analysis, which can make (18) hold, i.e., $\|\eta(i,j)\|_2 \leq \tilde{b}_\eta \gamma_{\max}$ and ensure $1 - \lambda_{\min}(\Omega)/\lambda_{\max}(\Xi) \neq \gamma_{\max}$). Therefore, we denote $\gamma_{\max} = \max\{1 - \lambda_{\min}(\Omega)/\lambda_{\max}(\Xi), \tilde{\gamma}_{\max}\}$ and $\gamma_{\min} = \min\{1 - \lambda_{\min}(\Omega)/\lambda_{\max}(\Xi), \tilde{\gamma}_{\max}\}$, which satisfy $0 \leq \gamma_{\min} < \gamma_{\max} < 1$. Then by again considering (22), we can further derive

$$V(e(i,j)) \leq [1 - \lambda_{\min}(\Omega)/\lambda_{\max}(\Xi)]^l V(e(i,0)) + b_\eta \sum_{l=0}^{j-1} [1 - \lambda_{\min}(\Omega)/\lambda_{\max}(\Xi)]^l \gamma_{\max}^{j-l}$$

$$\leq \lambda_{\max}(\Xi)b_0^2 \gamma_{\max}^2 + b_\eta \sum_{l=0}^{j-1} \gamma_{\max}^{j-1-l} \gamma_{\min}$$

$$\leq \lambda_{\max}(\Xi)b_0^2 \gamma_{\max}^2 + b_\eta \gamma_{\max} (1 - (\gamma_{\min}/\gamma_{\max})^2)/(\gamma_{\max} - \gamma_{\min})$$

$$\leq \lambda_{\max}(\Xi)b_0^2 \gamma_{\max}^2 + \gamma_{\max}^2 (1 - (\gamma_{\min}/\gamma_{\max})^2)/(\gamma_{\max} - \gamma_{\min})$$

where $V(e(i,0)) \leq \lambda_{\max}(\Xi)\|e(i,0)\|_2^2 \leq \lambda_{\max}(\Xi)b_0^2$ is also inserted. By the fact that $V(e(i,j)) \geq \lambda_{\min}(\Xi)\|e(i,j)\|_2^2$, we can use (23) to obtain $\|e(i,j)\|_2 \leq \sqrt{V(e(i,j))/\lambda_{\min}(\Xi)} \leq b_c \sqrt{\gamma_{\max}^2}$, where $b_c$ is bounded and given by $b_c = \sqrt{[\lambda_{\max}(\Xi)b_0^2 (\gamma_{\max} - \gamma_{\min})]/[\lambda_{\max}(\Xi) (\gamma_{\max} - \gamma_{\min})]}$. Due to $0 \leq \gamma_{\max} < 1$ (thus, $0 \leq \sqrt{\gamma_{\max}^2} < 1$) and the boundedness of $b_c$, we can conclude that $\lim_{j \to \infty} e(i,j) = 0$ holds with its limit being approached exponentially fast under the condition (9) if $G(i)$ has a spanning tree.

**Step (iii). We will show the exponential convergence of the formation objective (2).**

By induction, we can easily conclude based on the above Steps (i) and (ii) that $\lim_{j \to \infty} e(i,j) = 0$ holds with its limit being approached exponentially fast for all $i \in \mathbb{Z}_{N-1}$. This, together with the result (a) of Lemma 1, guarantees that the formation objective (2) holds with its limit being approached exponentially fast under the condition (9) if $G(i)$ has a spanning tree.
Remark 2 By the above proof, we have shown that the spanning tree condition on \( G(i) \) indeed provides a sufficient guarantee to the exponential convergence of the relative formation objective (2) which does not have any reference as prior knowledge for the agents. This condition is clearly more relaxed than the exponential convergence condition given in [13] for the formation objective (2), since in [13] it requires that there exists at least one vertex which is directly connected to all the others in \( G(i) \) (see Theorem 2 of [13] for details). In addition, it can be easily seen from the proof of Theorem 1 that this further result is achieved also by taking advantage of the Lyapunov analysis and the inductive approach.

Corollary 1 Consider the following 2-D Roesser system over \( i \in \mathbb{Z}_{N-1} \) and \( j \in \mathbb{Z}_+ \) [13]:

\[
\begin{bmatrix}
X^h(i + 1, j) \\
X^v(i, j + 1)
\end{bmatrix} =
\begin{bmatrix}
A_{11}(i) & A_{12}(i) \\
A_{21}(i) & A_{22}(i)
\end{bmatrix}
\begin{bmatrix}
X^h(i, j) \\
X^v(i, j)
\end{bmatrix}
\]

Where \( X^h(i, j) \in \mathbb{R}^p \) and \( X^v(i, j) \in \mathbb{R}^n \) are the states, \( A_{11}(i), A_{12}(i), A_{21}(i) \) and \( A_{22}(i) \) are real matrices of appropriate dimensions, and

(I) \( X^h(0, j) = 0 \) for \( j \in \mathbb{Z}_+ \) and \( X^v(i, 0) \) is bounded for \( i \in \mathbb{Z}_{N-1} \);

(II) \( A_{22}(i) \) is a stochastic matrix associated with \( G^{sl}(i) \), and \( A_{12}(i)1_n = 0 \).

Then there exists \( \alpha^v(i) \in \mathbb{R} \) such that

\[
\lim_{j \to \infty} \begin{bmatrix}
X^h(i, j) \\
X^v(i, j)
\end{bmatrix} =
\begin{bmatrix}
0 \\
\alpha^v(i)1_n
\end{bmatrix}
\]

holds with its limit being approached exponentially fast if and only if the directed graph \( G(i) \) has a spanning tree for \( i \in \mathbb{Z}_{N-1} \).

Proof This proof can be developed by considering Theorem 9 of [13] and following the same steps used in the proof of Theorem 1, and thus is omitted here.

We can clearly observe that Corollary 1 further improves the convergence result of Theorem 9 of [13] to converge exponentially fast. Also, this corollary improves the exponential convergence result of Theorem 10 of [13] by presenting a much relaxed condition which gives a necessary and sufficient guarantee to the convergence problem considered for networked 2-D Roesser systems.

4 Simulation Results

In this section, we again consider the numerical example of [13] and only perform the simulation in the Case (2)\(^1\). The nonzero learning gains of the formation algorithm (3) are chosen according to the condition (9) such that \( \phi_{21}(i) = \phi_{32}(i) = \phi_{43}(i) = 1.3 \) for \( i \in \{0, 1, 2, 3, 4, 20\}; \phi_{14}(i) = \phi_{21}(i) = \phi_{32}(i) = 0.26 \) for \( i \in \{5, 6, 7, 8, 9\}; \phi_{14}(i) = \phi_{21}(i) = \phi_{43}(i) = 0.52 \) for \( i \in \{10, 11, 12, 13, 14\}; \phi_{14}(i) = \phi_{32}(i) = \phi_{43}(i) = 0.39 \) for \( i \in \{15, 16, 17, 18, 19\} \). Here, we will apply \( e(i, j) \) defined in Lemma 1 to evaluate the coordination learning performance of the multi-agent system, where \( F(i) \) given in (7) is used without loss of generality.

\(^1\)For the detailed parameters used in this case, see Section 6 of [13].
We depict the simulation test results in Figs. 1 and 2. In Fig. 1, we describe processes of both the states of four agents (left) and relative formations between agents (right) along the time axis $i$ achieved after $j = 200$ iterations. It can be easily seen from Fig. 1 that the relative formations between agents are well achieved even without any reference prescribed as prior knowledge for the agents. Fig. 2 describes the process of the formation error evaluated by $\max_{0 \leq i \leq 19} \|e(i,j)\|_2$. Clearly, the coordination learning process for the multi-agent system converges along the iteration axis. By comparing $\max_{0 \leq i \leq 19} \|e(i,j)\|_2$ with $300 \times 0.95^j$, Fig. 2 also shows that the coordination learning process is guaranteed to have an exponentially fast convergence speed with the increase of the iteration number. Obviously, this simulation result can demonstrate the theoretical studies of Theorem 1.

5 Conclusions

In this paper, we have presented a further result for the coordination learning problem discussed by [13] in the case where there is not any reference as prior knowledge for the agents to accomplish the desired relative formations between them. We have proved that we can establish a necessary and sufficient condition for the exponential convergence of the considered formation problem of
agents. This not only improves the result of Theorem 2 in [13] but also can provide insights into the development of a necessary and sufficient condition for the exponential convergence of the general networked 2-D Roesser systems considered in [13].

References