An Area-based Approach to Ranking Fuzzy Numbers in Fuzzy Decision Making

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Abstract

This paper presents an area-based approach to ranking fuzzy numbers in fuzzy decision making. To ensure that all the information that a fuzzy number has is adequately considered, the concepts of the absolute area and the degree of deviation of a fuzzy number are integrated into the process of comparing and ranking fuzzy numbers. To help the decision maker better address the risk inherent in the decision making process, the attitude of the decision maker towards risk is considered in developing the overall ranking index for each fuzzy number. As a result, the proposed approach can adequately address the problems that existing approaches suffer from. Examples are presented that shows the proposed approach is effective and efficient in comparing and ranking fuzzy numbers due to its rationality in concept, simplicity in computation, and discriminatory ability in differentiating similar fuzzy numbers.

Keywords: Ranking Fuzzy Numbers; Fuzzy Decision Making; Absolute Area; Degree of Deviation

1. Introduction

Comparing and ranking fuzzy numbers is an important part of fuzzy decision making. This is because these fuzzy numbers can often be obtained in a fuzzy decision making situation to represent the overall utilities of decision alternatives, commonly referred to as fuzzy utilities [1,6,13]. Most fuzzy decision models developed in the context of multi-attribute utility theory consist of two phases including the fuzzy utility aggregation process and the fuzzy utility comparison process [13]. A comparison between fuzzy utilities thus is a comparison between decision alternatives which is critical for effective decision making under uncertainty.

Comparing and ranking fuzzy numbers are complex and challenging. This is because fuzzy numbers usually represented by possibility distributions can often overlap each other in many practical situations [2,13]. It is difficult, if not impossible, to determine clearly whether one fuzzy number is larger or smaller than another, in particular when these two fuzzy numbers are similar to each other. As a result, comparing and ranking fuzzy numbers becomes a critical problem to be solved in fuzzy decision making [2,11,18].

Numerous approaches have been developed for comparing and ranking fuzzy numbers in the literature. Bortoland and Degani [1], Nakamura [17], Lee and Li [4] [15], Tseng et al. [16], and Chen and Hwang [2], Chen and Lu [18], Yeh and Deng [13] give an intensive investigation of existing ranking approaches from

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different perspectives. These approaches can be classified into four major categories, including (a) independent ranking, (b) reference set based ranking, and (c) pairwise comparison based ranking, and (d) linguistic approximation. Existing approaches have addressed the problem of comparing and ranking fuzzy numbers from different perspectives with sounding outcomes [2]. These existing approaches, however, are not totally satisfactory due to the lack of discrimination ability in differentiating similar fuzzy numbers, the sophistication of the approach, the inconsistent ranking outcomes under some circumstances, the counter-intuitive rankings produced, and the considerable computational effort required [2, 13].

To effectively address the problems as above, this paper presents an area-based approach for comparing and ranking fuzzy numbers. Based on a comprehensive review of the area-based approaches for comparing and ranking fuzzy numbers, the concepts of the absolute area and the degree of deviation of a fuzzy number are integrated in developing the approach so that all the information that a fuzzy number has is effectively considered. To appropriately tackle the risk inherent in fuzzy decision making, the attitude of the decision maker towards risk is considered in determining the overall performance index value of a fuzzy number. As a result, the proposed approach can adequately address the problems that existing approaches suffer from. Examples are presented that shows that the proposed approach is effective and efficient in comparing and ranking fuzzy numbers due to its rationality in concept, simplicity in computation, and discriminatory ability in differentiating similar fuzzy numbers.

2. Basic Concepts of Fuzzy Numbers

Much decision making in the real world takes place in an environment where the goals, the constraints, and the consequences of possible actions are not known precisely. As a result, the uncertainty associated with the decision making process cannot be adequately tackled under the promise of randomness using existing tools such as the probability theory [2]. Conventional approaches such as statistical analysis and game theory cannot adequately handle the fuzziness inherent in human behavior.

The application of fuzzy sets theory in decision analysis provides an effective means for tackling the fuzziness inherent in the human decision making process. Fuzziness arising from a lack of clearly defined boundaries between groups of elements in terms of ambiguity or vagueness in mathematical models of empirical phenomena can be better handled [13]. In general a fuzzy set can be defined as follows:

**Definition 1.** A fuzzy set $A$ of the universe of discourse $U$ is defined by a membership function $\mu_A: U \rightarrow [0,1]$, where $\mu_A(x)$ is the degree of membership of $x$ in $A$, and $[0,1]$ is the closed unit interval on the real line $\mathbb{R}$. Very often, fuzzy set $A$ of $U$ can be expressed as

$$A = \left\{ \frac{\mu_A(x)}{x}, x \in U, \mu_A(x) \in [0, 1] \right\}$$

(1)

**Definition 2.** A fuzzy set $A$ of the universe set $U$ is normal iff $\sup_{x \in U} \mu_A(x) = 1$.

**Definition 3.** A fuzzy set $A$ of the universe set $U$ is convex iff

$$\mu_A(\lambda x + (1-\lambda)y) \geq \mu_A(x) \wedge \mu_A(y), \forall x, y \in U, \forall \lambda \in [0,1],$$

where $\wedge$ denotes the minimum operator.
A fuzzy number $A$ is a convex fuzzy set characterized by a given interval of real numbers, each with a grade of membership between 0 and 1. Its membership function is piecewise continuous, and satisfies the following conditions:

(a) $\mu_A(x) = 0$ for each $x \in (-\infty, a] \cup [d, +\infty)$,

(b) $\mu_A(x) = f_A^L(x)$ is non-decreasing on $[a, b]$ and $\mu_A(x) = f_A^R(x)$ non-increasing on $[c, d]$,

(c) $\mu_A(x) = 1$ for each $x \in [a, b]$.

where $f_A^L(x): [a, b] \rightarrow [0, 1]$ and $f_A^R(x): [c, d] \rightarrow [0, 1]$ are two continuous mappings to the closed interval $[0, 1]$. In this definition $f_A^L(x)$ is a non-decreasing function referred to as the left membership function, and $f_A^R(x)$ is a non-increasing function referred to as the right membership function. The inverse functions of $f_A^L(x)$ and $f_A^R(x)$ both exist, respectively denoted by $g_A^L(x): [0, 1] \rightarrow [a, b]$ and $g_A^R(x): [0, 1] \rightarrow [c, d]$.

3. Area-based Approaches to Comparing and Ranking Fuzzy Numbers

Comparing and ranking fuzzy numbers in fuzzy decision making is usually based on the relative position of these fuzzy numbers in a two-dimensional space [2]. There are various ways to determine the relative positions of these fuzzy numbers dependent on individual ranking approaches used. One of the common ways is to calculate the area that a fuzzy number covers with respect to specific reference sets for developing the overall ranking index. Typical approaches in this regard include Chu and Tsao [5], Deng et al. [6], Wang and Liu [9], Nejad [10], and Wang and Luo [12]. Fig.1 presents an overview of how the area of a fuzzy number is calculated in these approaches for comparing and ranking fuzzy numbers.

Chu and Tsao [5] propose a ranking approach based on comparing the common area of fuzzy numbers between the centroid point and the original point in each fuzzy number as shown in Fig.1(a). The geometric centre represented by the centroid point for each fuzzy number is determined based on Cheng [3]. The area between the centroid point and the original point of a fuzzy number is calculated using the Hamming distance [2]. Such an area becomes the basis for comparing and ranking the fuzzy numbers concerned. This approach is simple in concept, and the computation involved is effective when the fuzzy numbers involved
are triangular or trapezoidal fuzzy numbers. The approach, however, may lead to counter-intuitive ranking outcomes under some circumstances [8, 10], especially when comparing the abnormal fuzzy numbers. This may be due to the fact that the approach does not consider the dispersion of each fuzzy number in comparing the fuzzy numbers [2].

Deng et al. [6] present a revised approach for comparing and ranking fuzzy numbers as shown in Fig. 1 (b). Such an approach focuses on the radius of the gyration point instead of the centroid point. Similar to the approach of Chu and Tsao [5], this approach treats x axis and y axis as the same degree of importance in the ranking process. Furthermore, these two approaches consider only the mean value of each fuzzy number. The deviation of a fuzzy number usually reflected by the dispersion of a fuzzy number [2] is not taken into account in the ranking process. As a result, counter-intuitive ranking outcomes may be present when comparing abnormal fuzzy numbers.

Wang and Liu [9] develop an area-based approach based on the degree of deviation of a fuzzy number with respect to the maximizing set and the minimizing set [14]. In particular the left degree of deviation, the right degree of deviation, and a transfer coefficient of a fuzzy number as shown in Fig.1(c) are introduced in ranking the fuzzy numbers based on the rationale that a fuzzy number is preferred if it is closer to the maximizing set and at the same time is farther away from the minimizing set. This approach, however, is still unsatisfactory in comparing and ranking fuzzy numbers under some circumstances. For example, incorrect ranking outcomes may happen when the left degree of deviation, the right degree of deviation, or a transfer coefficient of a fuzzy number is zero. This is due to the fact that this approach fails to consider the absolute position of each fuzzy number in determining the ranking index.

To overcome the shortcomings of the approach of Wang and Liu [9], Nejad [10] proposes an approach for comparing and ranking fuzzy numbers as shown in Fig.1(d). Such an approach introduces two special fuzzy numbers as the reference sets to the existing set of fuzzy numbers in comparison for preventing the left degree of deviation, the right degree of deviation, or a transfer coefficient of a fuzzy number from attaining zero. As a result, the problems mentioned above are avoided in comparing and ranking fuzzy numbers. This approach, however, suffers from similar problems as those of Wang and Liu because it also does not consider the absolute position of the fuzzy numbers in comparison.

Wang and Luo [12] develop a ranking approach based on the positive and negative ideal points as shown in Fig.1(e) for comparing and ranking fuzzy numbers. The positive and negative ideal points are respectively defined as 

\[ x_{max} = \sup \bigcup_{i=1}^{N} x_i \] and 
\[ x_{min} = \inf \bigcup_{i=1}^{N} x_i \]

where \( x_i = \{ x \mid \mu_A(x) > 0 \} \). They are used as reference sets so that the left area \( S_L(i) \) and the right area \( S_R(i) \) relative to the reference sets can be calculated as

\[ S_L(i) = (b_i - x_{min}) - \int_{a_i}^{b_i} f_A(x)dx \quad , \quad S_R(i) = (x_{max} - c_i) - \int_{c_i}^{d_i} f_A(x)dx \]  \hspace{1cm} (2)\]

By considering the attitude of the decision maker towards risk represented by alpha \((0 \leq \alpha \leq 1)\), the overall ranking index for each fuzzy number can be determined by

\[ RIA(A_i) = \frac{1}{2} \left[ (\frac{S_L(i)}{x_{max} - x_{min}})r_L(i) + (1 - \frac{S_R(i)}{x_{max} - x_{min}})r_R(i) \right] \]  \hspace{1cm} (3)\]
where \( r_L(i) \) and \( r_R(i) \) are the left and the right risk factors calculated by

\[
\begin{align*}
    r_L(i) &= 1 + (a - 0.5) \frac{b_i - a_i}{x_{\text{max}} - x_{\text{min}}}, \\
    r_R(i) &= 1 + (\alpha - 0.5) \frac{d_i - c_i}{x_{\text{max}} - x_{\text{min}}},
\end{align*}
\]

This approach is proved to be effective in comparing and ranking fuzzy numbers under some situations. The approach, however, still suffers from several shortcomings. For example, the approach can compare and rank only the fuzzy numbers in a same set. It cannot be used for comparing and ranking fuzzy numbers from different sets which is often critical in fuzzy decision making [2]. Furthermore, this approach cannot adequately handle the attitude of the decision maker towards risk in fuzzy decision making which is often required [13].

The area-based approaches above have shown their merits in comparing and ranking fuzzy numbers from different perspectives. They, however, still suffer from some common shortcomings due to the inadequacy of considering the attitude of the decision maker towards risk and the ignorance of the absolute position of the fuzzy number in all the fuzzy numbers. To overcome these shortcomings, the next section presents a new area-based approach based on the absolute area and the degree of deviation of a fuzzy number for adequately comparing and ranking fuzzy numbers through effectively considering the absolute position of a fuzzy number and the attitude of the decision maker towards risk.

4. An New Area-Based Approach

This section presents a new area-based approach for comparing and ranking fuzzy numbers. The approach makes a full use of the information associated with each fuzzy number based on the absolute position of a fuzzy number, the degree of deviation, and the attitude of the decision maker toward risk in the decision making process. To facilitate the development of the approach, these three concepts are discussed first in the following.

The absolute area of a fuzzy number describes the position of each fuzzy number in the real line. It is the overall mean area of fuzzy number for showing the absolute mean position of a fuzzy number on the real line \( R \). To facilitate determining the absolute position of a fuzzy number, the left and right spreads of a fuzzy number are used and signed area index of it is adopted here. The absolute area of a fuzzy number is defined in the following.

**Definition 4.** The absolute area of fuzzy number \( A \) is determined by

\[
\text{Ab}_-\text{area}(A) = \left( \int_{\lambda} g_A^R(\lambda)d\lambda \right) + \left( \int_{\lambda} g_A^L(\lambda)d\lambda \right) / 2 = \text{(S1 + S2 + S3 + S4)}/2 \quad (5)
\]

Where \( S1 = \left( \int_{\lambda} g_A^R(\lambda)d\lambda \right) \), \( S2 = \left( \int_{\lambda} g_A^L(\lambda)d\lambda \right) \), \( S3 = \left( \int_{\lambda} g_A^R(\lambda)d\lambda \right) \), \( S4 = \left( \int_{\lambda} g_A^L(\lambda)d\lambda \right) \). \text{Ab}_-\text{area}(A) can be proper as the main part of the overall index value of a fuzzy number \( A \) according to the absolute position of each fuzzy number, which is sometimes
needed to be modified when considering different attitudes of the decision maker towards risk. It is the basis of the comparability of all the fuzzy numbers across different sets due to its independence.

The degree of deviation of a fuzzy number is often used for reflecting the attitude of the decision maker towards risk in fuzzy decision making [2,15,16]. It is usually determined by the spread, the deviation basis and the deviation coefficient of a fuzzy number. The deviation basis is usually relevant to only the area of a fuzzy number determined by

$$ S(A) = \int_0^1 (g^b_A(\lambda) - g^l_A(\lambda))d\lambda $$

(6)

To control the relative deviation degree of all the fuzzy numbers across different sets reasonably and fairly in the practical approach, a deviation scale $\lambda$ acted as the denominator of the deviation coefficient which is applied to all the fuzzy numbers from different sets is introduced. For this purpose, the concept of the ideal solution is used because (a) it can obtain a uniform deviation reference for each fuzzy number from different sets, and (b) it is easy to obtain two uniform and simple crisp real numbers as the deviation scale boundaries. The deviation scale of all fuzzy numbers from different sets can be determined as follows

$$ \lambda^* = \begin{cases} x^*_{\text{max}} - x^*_{\text{min}}, & x^*_{\text{max}} - x^*_{\text{min}} > 1 \\ 1, & x^*_{\text{max}} - x^*_{\text{min}} < 1 \end{cases} $$

(7)

where $x^*_{\text{max}} = \sup \bigcup_{j=1}^{m_j} x^j_{\text{max}}, x^*_{\text{min}} = \inf \bigcup_{j=1}^{m_j} x^j_{\text{min}}, x^j_{\text{max}} = \sup \bigcup_{i=1}^{n_j} S(A^j_i) , x^j_{\text{min}} = \inf \bigcup_{i=1}^{n_j} S(A^j_i)$. The deviation coefficient is inverse ratio in it. The left and right deviation range of a fuzzy number are always given by

$$dv_l(A) = \int_{a_l}^{b_l} f^l_A(x)dx, dv_r(A) = \int_{c_l}^{d_l} f^r_A(x)dx$$

(8)

Based on the deviation basis, the deviation coefficient and the deviation range, simultaneously considering the different impact degree and the different sign of left and right deviation, the relative degree of deviation of fuzzy number $A^j_i$ can be calculated by

$$R_{\text{Dev}}(A^j_i) = \begin{cases} 0, & \alpha = 0.5 \\ S(A^j_i) \left(\alpha dv_r(A^j_i) - (1-\alpha)dv_l(A^j_i)\right), & \alpha \neq 0.5, \alpha \in [0,1] \end{cases}$$

(9)

Where $\alpha \in [0,1]$ is used for reflecting the attitude of the decision maker towards risk in fuzzy decision making [13]. With the determination of the absolute area and the degree of deviation of a fuzzy number, an overall performance index for each fuzzy number can be by

$$P(A^j_i) = Ab_{\text{area}}(A^j_i) + R_{\text{Dev}}(A^j_i)$$

(10)

The larger the preference index value is, the more preferred the fuzzy number is. Obviously, it can be used to compare and rank all the fuzzy numbers cross different sets and with different attitude of decision making towards risk.

Summarizing the discussion above, the new area-based approach for comparing and ranking fuzzy
numbers can be presented as an algorithm as follows:

**Step 1.** Determine the uniform deviation scale $\lambda$ by (7).

**Step 2.** Determine the $\alpha$ value based on the attitude of the decision maker towards risk.

**Step 3.** Calculate the absolute area $A_{bare A_{ij}}$ of fuzzy number $A_{ij}$ by (5).

**Step 4.** Determine the deviation basis $S(A_{ij})$, the left deviation area $dv_L(A_{ij})$, and the right deviation area $dv_R(A_{ij})$ of fuzzy number $A_{ij}$ by (6) and (8).

**Step 5.** Calculate the degree of deviation $R_{Dev}(A_{ij})$ of fuzzy number $A_{ij}$ by (9).

**Step 6.** Calculate the overall preference index $P_i(A_{ij})$ of fuzzy number $A_{ij}$ by (10).

**Step 7.** Rank the fuzzy numbers in the descending order of the value of $P_i(A_{ij})$.

5. **Comparative Examples**

The area-based approach presented above shows some important advantages over other comparable approaches in terms of the computational simplicity, the rationality, and the discriminatory ability in differentiating similar fuzzy numbers. With the triangular fuzzy numbers widely adopted in most practical applications, the computation involved in the proposed approach is simple and straightforward. All the typical examples of the fuzzy numbers presented in the literature relevant to these approaches are examined. The examination result shows that the proposed approach can always obtain satisfactory outcomes, especially under the situation in which the consideration of the attitude of the decision maker towards risk is required in fuzzy decision making.

To demonstrate the merits of the proposed approach, several examples are presented in this section for conducting some comparative studies with those area-based methods in section 3. These comparative studies show the satisfactory outcomes of the proposed approach, simultaneously, have their own purpose. Example 1 shows mainly the distinct role of the concept of the absolute area of a fuzzy number for ranking the fuzzy numbers from different sets. Example 2 gives a better outcome comparing with the approach of Wang and Luo [12] while considering the attitude of the decision maker towards risks. The last example demonstrates the appropriateness of the proposed approach in comparing and ranking fuzzy numbers with satisfactory ranking outcomes.

**Example 1.** Consider two sets of fuzzy numbers as shown in Fig.2 (a).

- **Set 1.** $A1=(0.5,0.5,1)$, $A2=(0.5,1,1.5)$,
- **Set 2.** $B1=(0,0.2,0.7)$, $B2=(0.1,2)$, $B3=(1.2,1.5,2)$.

Using the proposed approach, the ranking of these fuzzy numbers can be obtained. To facilitate the comparative analysis of the proposed approach with the comparable approaches discussed in Section 3, the ranking index value of each fuzzy number and their overall ranking are presented in the left half of Table1.
As shown in the left half of Table 1, the ranking result $A_1 < A_2$ in Set 1 and $B_1 < B_2 < B_3$ in Set 2 with the different attitudes of the decision maker towards risks can be obtained with the proposed approach, which is the same as intuition in every set alone. Unsatisfactory ranking outcomes, however, may be present when these five fuzzy numbers are required to be compared all together across the two sets with the approach of Wang and Luo [12] or other area-based approaches discussed in Section 3. Only the proposed approach can effectively address this problem.

**Example 2.** Consider two sets of trapezoidal fuzzy numbers as follows:

**Set 1.** $A_1=(0.2,0.4,0.6,0.8)$ and $B_1=(0.3,0.3,0.6,0.8)$ as shown in Fig. 2 (b),

**Set 2.** $A_2=(0.6,0.8,1)$, $B_2=(0.7,0.8,0.98)$, $C_2=(0.7,0.8,0.9)$ as shown in Fig. 2 (c).

The right half of Table 1 shows the ranking index value of each fuzzy number and their overall ranking with the proposed approach. Intuitively, the fuzzy number $B_1$ of Set 1 is not smaller than $A_1$ no matter regardless of the attitude of the decision maker towards risks. With the use of the approach of Wang and Luo [12], the ranking order is $A_1 > A_2$. This is evidently incorrectly as the reasonable ranking should be at $A_1 \sim A_2$ obtained with the proposed approach.

**Example 3.** Consider the three sets of fuzzy numbers from Wang and Liu [9] and Yao and Wu [11] shown in Fig.3(a), Fig.3(b) and Fig.3(c) respectively.

**Set 1.** $A_1=(6,6,1,1)_{LR}$, $A_2=(6,6,0,1)_{LR}$ and $A_3=(6,6,0,1)_{LR}$;

**Set 2.** $B_1=(0.5,0.5,0.1,0.5)_{LR}$, $B_2=(0.7,0.7,0.3,0.3)_{LR}$ and $B_3=(0.9,0.9,0.5,0.1)_{LR}$;

**Set 3.** $C_1=(0.4,0.7,0.1,0.2)_{LR}$, $C_2=(0.7,0.7,0.4,0.2)_{LR}$ and $C_3=(0.7,0.7,0.2,0.2)_{LR}$.

Table 2 shows the ranking index value and the ranking of each fuzzy number in Example 3 with the
proposed approach and the approach of Wang and Luo [12] with respect to the different attitude of the decision maker towards risks in fuzzy decision making. It is evident that the proposed approach can always obtain satisfactory ranking outcomes while there are counter-intuitive ranking outcomes with the approach of Wang and Luo [12]. In particular, the proposed approach can compare fuzzy numbers across the three sets without producing counter-intuitive ranking outcomes. This is a significant advantage in fuzzy decision making when the evaluation and selection of decision alternatives has to be conducted across several groups across the organization or even across various organizations. Furthermore, we can see that the outcomes of proposed approach are same as most of those area-based methods presented in section 3 in every set alone.

\[ y = x \]

6. Conclusion

This paper proposes a new area-based ranking approach to comparing and ranking fuzzy numbers based on the concepts of the absolute area and the degree of deviation of a fuzzy number. With the examples presented above, the proposed approach is demonstrated to be simple and effective in ranking fuzzy numbers, especially when triangular fuzzy numbers are present. This is because the proposed approach has made a full use of all the information of a fuzzy number while incorporating the attitude of the decision maker towards risks. Furthermore, the adequate consideration of the absolute position of a fuzzy number allows the comparison of all the fuzzy numbers from different groups without compromising the ranking. Such a merit of the proposed approach is of practical significance in fuzzy decision making in which the evaluation and selection of decision alternatives across various groups is required.

<table>
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<th>A2</th>
<th>A3</th>
<th>Ranking</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>Ranking</th>
<th>C1</th>
<th>C2</th>
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<td>6.287</td>
<td>A1=A2=A3</td>
<td>0.611</td>
<td>0.707</td>
<td>0.802</td>
<td>B1=B2=B3</td>
<td>0.582</td>
<td>0.655</td>
<td>0.703</td>
<td>C1=C2=C3</td>
<td>C1=C2=C3=C3=B3=A1=A2=A3</td>
</tr>
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References