Performance Analysis of a Compact Support Radial Basis Function Based on Thin-Plate Splines

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Abstract

In landmark-based registration, radial basis function-based transformation play an important role. The compact support radial basis function based on thin-plate splines (CSTPS) is an effective function which has been used to perform interpolation in elastic registration of medical images. In this paper, the positive definite property of CSTPS is theoretically proved. Thus, the solvability of the equations determining the closed-form coefficients for the transformation functions is guaranteed. Next, the similarity between CSTPS with large support and TPS is proved based on a simple point-matching model. This behavior is advantageous for providing overall smooth deformation. Moreover, experiments involving transformations on random point sets and medical images demonstrated that the registration accuracy of CSTPS is satisfactory and stable for both local and global deformations.

Keywords: Landmark-based Image Registration; Image Transformation; Compact Support Radial Basis Function.

1. Introduction

In landmark-based image registration, spatial transformation plays an important role. Transformations are applied to the source image so that it aligns with the target. Image transformations based on the basis function expansion model deformations using a set of basis functions. The coefficients are adjusted so that the combination of basis functions fits the displacement field [1]. Various compact support radial basis function-based methods [4–10] have been proposed for local deformations. Siddiqui [4] used locally constrained cosines as a radial basis function which was used for elastic image registration and improved registration quality. Fornefett [6] used Wendland \( \psi \) -functions [9], which have a parameter to control their locality, to perform image interpolation. Wu [10] and Buhmann [8] proposed a larger class of smooth radial functions with compact support for interpolation. More and more, elastic image registration with local deformation regions is being used in medical image registration.

Recently, a novel compact support radial basis function based on thin-plate splines (CSTPS) has been proposed [7]. From extensive experience, it has been observed that CSTPS has outstanding performance in implementing transformations, including low bending energy for local deformation measured over the whole deformation field, which results in small image-registration errors. In addition, it has been found that the deformation results using CSTPS with large support approach closely those achieved using TPS. However, the properties of CSTPS, such as the positive definite property, registration accuracy and the

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similarity between CSTPS and TPS, have not been analyzed. The major contribution of this paper is to prove theoretically the positive definite property of CSTPS and the similarity between CSTPS with large support and TPS based on a simple point-matching model. Moreover, the registration accuracy of CSTPS is evaluated here in detail, including both linear and nonlinear deformations.

2. Compact Support Radial Basis Function Based on Thin-plate Splines and Solvability

Compact support RBFs (CSRBFs) have advantages of local influence of a landmark pair on the registration result. Zhang [7] proposed a novel CSRBF, based on thin-plate splines, named CSTPS. \phi_{CSTPS} = e\left(r^2 \ln r^2 + 1/e\right), \quad 0 < r \leq 1/\sqrt{e} .

CSTPS is a compactly supported decreasing function and can be used for local deformation interpolation.

The transformation function using compact support radial basis functions is

\[ f(x) = x + \sum w_i \varphi\left(\|x - p_i\|/c\right), \]

where \( c \) is the support of \( \varphi \) which controls the influence of each landmark. The coefficients \( W = (w_1, w_2, \ldots, w_N)^T \) are solved using the linear equations \( Kw = Q - P \). \( P \) and \( Q \) denote a set of \( N \) source and target landmarks respectively. \( K \) denotes the \( N \times N \) matrix given by

\[ K_{ij} = \varphi\left(\|p_i - p_j\|/c\right). \]

To ensure the solvability of the equation \( Kw = Q - P \), the matrix \( K \) should be nonsingular in all cases where \( p_i \neq p_j \). If \( \varphi \) is a positive definite function, \( K \) can be proved to be invertible [10], and the solvability of the equations is ensured. Let \( CS \) denote the set of compactly supported functions in one dimension. A function \( \varphi \) which is positive definite in \( d \) dimensions can be written as \( \varphi \in PD_d \). Assume that the symmetric function \( \varphi \) has compact support and that the support is contained in \([-1,1]\). Wu [10] presented the following theorem:

**Theorem.** A univariate function \( \varphi(r) \) is in \( CS \cap PD_1 \) iff \( \int_0^1 \varphi(x) \cos(\omega x) dx \geq 0 \).

For CSTPS, the support is \([0,1/\sqrt{e}]\). Let \( \approx \) denote equality up to a positive constant factor. Then

\[ \int_0^{1/\sqrt{e}} \varphi_{CSTPS}(x) \cos(\omega x) dx \approx -\frac{4}{\omega^2} \sin\left(\frac{\omega}{\sqrt{e}}\right) + \frac{4}{\omega^3} \int_0^{1/\sqrt{e}} \sin x \frac{x}{x} dx . \]

Let \( f(\omega) = -\sin \omega + \int_0^\omega \sin x dx \). The first derivative of \( f(\omega) \) on \([0,\pi]\) is \( f'(\omega) = -\cos \omega + \frac{\sin \omega}{\omega} \). Using the series expansions of \( \cos \omega \) and \( \sin \omega \), \( f'(\omega) \) can be represented as

\[ f'(\omega) = \sum_{k=0}^{\infty} \frac{\omega^{2k}}{(4k+3)!} \left[ 4k + 2 - \frac{1}{4k + 5} \omega^2 \right] . \]

For \( \omega \in [0,\pi] \), it can be easily derived that \( 4k + 2 - \frac{1}{4k + 5} \omega^2 > 0 \) for \( k \in Z \). Therefore, \( f'(\omega) > 0 \). This means that the function \( f(\omega) \) is a strictly monotonic increasing function. Because
\( f(0) = 0 \), then \( f(\omega) > 0 \) for \( \omega \in [0, \pi] \). On the other hand, \( \int_0^\pi \frac{\sin x}{x} \, dx > 1 \) for \( \omega > \pi \). In other words, \( f(\omega) > 0 \) for \( \omega > \pi \). In summary, \( f(\omega) > 0 \) for \( \omega \in [0, \infty) \). Correspondingly, it can be proved that \( \int_0^{\pi/2} \phi_{\text{CSTPS}}(x) \cos(\omega x) \, dx \geq 0 \) for the compact support function CSTPS. Therefore, CSTPS is positive definite in one-dimensional space.

3. Comparison with the TPS

Zhang [7] observed the deformation results using the CSTPS with large support \( c \) are similar to the TPS very much. But Zhang did not prove the similarity in theory. Next, we will prove the similarity based on a simple point-matching modal.

Suppose four source landmarks locate as a rhombus, and only the upper landmark is shifted upward \( \Delta \). The locations of the source landmarks are \( P = \{(0,1), (-1, 0), (0, -1), (1, 0)\} \), and the target landmarks are \( Q = \{(0,1+\Delta), (-1, 0), (0, -1), (1, 0)\} \). We only consider the interpolant function along \( y \) direction.

The interpolant function using \( \phi_{\text{CSTPS}} \) along \( y \) direction is

\[
 f_{\text{CSTPS}} = y + \sum_i w_i \phi_{\text{CSTPS}} \left( \frac{\|P_i - (x,y)\|}{c} \right).
\]

The coefficients are computed as

\[
 w_i = \frac{1 + \beta - 2\alpha^2}{(1 - \beta)(1 + \beta^2 - 4\alpha^2)} \Delta, \quad w_2 = \frac{\alpha}{(1 + \beta)^2 - 4\alpha^2} \Delta, \quad w_3 = \frac{-\beta^2 - \beta + 2\alpha^2}{(1 - \beta)(1 + \beta^2 - 4\alpha^2)} \Delta, \quad w_4 = w_2,
\]

where \( \alpha = \phi_{\text{CSTPS}}(\sqrt{2}/c), \beta = \phi_{\text{CSTPS}}(2/c) \). When the support \( c \) is very large, it can be supposed that \( \alpha \approx 1, \beta \approx 1 \). The following equations can be derived:

\[
 w_1 + w_3 \approx -(w_2 + w_4), \quad w_1 + w_2 \approx \frac{1}{2(1 - \beta)} \Delta,
\]

\[
 w_1 + w_4 = \frac{1 + \beta}{(1 + \beta)^2 - 4\alpha^2} \Delta, \quad w_2 - w_3 = \frac{1}{1 - \beta}, \quad \sum_i w_i \approx 0.25 \Delta. \quad (1)
\]

Considering the relation between \( \phi_{\text{CSTPS}} \) and \( \phi_{\text{TPS}} = r^2 \ln r^2, \phi_{\text{CSTPS}}(r) = 1 + e\phi_{\text{TPS}}(r) \), \( 0 < r \leq 1/\sqrt{e} \). The interpolant function using \( \phi_{\text{CSTPS}} \) can be represented as

\[
 f_{\text{CSTPS}}(x,y) = y + \sum_i w_i + e \sum_i w_i \phi_{\text{TPS}} \left( \frac{\|P_i - (x,y)\|}{c} \right). \quad \text{Let } g(x,y) = \sum_i w_i \phi_{\text{TPS}} \left( \frac{\|P_i - (x,y)\|}{c} \right),
\]

then

\[
 g(x,y) = \frac{w_1 + w_4}{2} \sum_{i=4} (-1)^{i+1} \phi_{\text{TPS}} \left( \frac{\|P_i - (x,y)\|}{c} \right)
 + \frac{w_1 - w_4}{2} \left( \phi_{\text{TPS}} \left( \sqrt{x^2 + (y-1)^2}/c \right) - \phi_{\text{TPS}} \left( \sqrt{x^2 + (y+1)^2}/c \right) \right) \quad (2)
\]
Bookstein [2] derived that
\[ \sum_{i=1}^{s} (-1)^{s+1} \varphi_{TPS} \left( \| P_i - (x, y) \| \right) = \frac{4 y^2 - 4 x^2}{x^2 + y^2 + 1} \cdot g(x, y) \]
is expressed as,
\[ g(x, y) = \frac{1 + \beta}{2 (1 + \beta)^2 - 4 \alpha^2} \frac{4 y^2 - 4 x^2}{x^2 + y^2 + 1} \frac{\Delta}{c^2} + \frac{\Delta}{2 (1 - \beta)} \left[ \varphi_{TPS} \left( \frac{\sqrt{x^2 + (y-1)^2}}{c} \right) - \varphi_{TPS} \left( \frac{\sqrt{x^2 + (y+1)^2}}{c} \right) \right] \quad (3) \]

When \( c \) is very large, it can be approximated that \( \ln \left( \frac{x^2}{c^2} \right) \approx \ln \left( \frac{1}{c^2} \right) \). Then \( \varphi_{TPS} \left( \frac{\sqrt{x^2 + (y-1)^2}}{c} \right) - \varphi_{TPS} \left( \frac{\sqrt{x^2 + (y+1)^2}}{c} \right) \approx -\frac{4 y}{c^2} \ln \frac{1}{c^2} \) and \( \frac{1}{2 (1 - \beta)} \left[ -\frac{4 y}{c^2} \ln \frac{1}{c^2} \right] \approx \frac{y}{2e} \). It can be derived that \( (1 + \beta)^2 - 4 \alpha^2 \approx \frac{16 e}{c^2} \ln 2 \). Considering that \( \beta \approx 1 \), it can be derived
\[ \frac{1}{2 (1 + \beta)^2 - 4 \alpha^2} \approx \frac{1}{16 e \ln 2} \approx \frac{0.09}{e} \]. Then \( g(x, y) \approx \frac{0.09}{e} \frac{4 y^2 - 4 x^2}{x^2 + y^2 + 1} \Delta + \frac{y}{2e} \), and
\[ f_{CSTPS} (x, y) \approx 0.25 \Delta + (1 + 0.5 \Delta) y + 0.09 \frac{4 y^2 - 4 x^2}{x^2 + y^2 + 1} \Delta \quad (4) \]

On the other hand, the interpolant function using \( \varphi_{TPS} \) along \( y \) direction is
\[ f_{TPS} (x, y) = 0.25 \Delta + (1 + 0.5 \Delta) y + 0.09 \frac{4 y^2 - 4 x^2}{x^2 + y^2 + 1} \Delta \quad (5) \]

It can be seen that the interpolant function using \( \varphi_{CSTPS} \) is similar to that using \( \varphi_{TPS} \). That implies the deformation results using CSTPS with large support would be similar to TPS. As we know, TPS performs relatively well when the images have global geometric differences. That means CSTPS with large support would have outstanding performance in global image registration. More importantly, when interpolating using CSRBFs with large support, the matrix \( K \) usually becomes ill-conditioned with an increasing number of landmarks. But a large number of experiments demonstrate that the matrices computed by CSTPS are stable and solvable. The reason possibly is the performances of CSTPS with large support are similar to TPS.

4. Registration Accuracy

Given a set of corresponding points in the reference and source images, a good transformation should not warp the source image too much and should not cause significant mis-registration away from the corresponding points [11]. We will compare the registration accuracy of several CSRBFs. The evaluated
CSRBFs include $\varphi_{p,3,1} = \left(1-r\right)^3 \left(4r+1\right)$, $\varphi_{w,1,0} = \left(1-r\right)^3 \left(1+3r+r^2\right)$, $\varphi_{Buh} = 1+3r^2-4r^3+r^3 \ln r^2$, and $\varphi_{Cos} = \left(1+\cos\left(r\pi\right)\right)/2$. $0 \leq r \leq 1$.

4.1. Local Deformations on Points in 2D Space

Fig.1(a) shows the deformation results for 30 points placed on a line with support $c=20$. Only four landmarks, marked by circles, are shifted upward by five points. An ideal transformation will not generate a nonlinear mapping of the line as long as the landmark correspondences do not contain nonlinear geometric differences. Fig.1(b) shows the deformation results for 30 points placed on a semicircle with support $c=30$. Seven landmarks are shifted by five points along the radius, as shown in the last row of Fig.1(b). The remaining points should be shifted similarly to preserve nonlinear geometric relations.

The Error and MaxOffset parameters were used to compare the deformation results. The Error parameter is the maximum distance between the target positions of any point from its expected position. The parameter MaxOffset is the maximum change of offset between two neighboring points [4]. It can be used to monitor the smoothness of the transformation functions. Table 1 contains a comparison of the deformation results illustrated in Fig.1(a) and Fig.1(b). It can be observed in Tables that CSTPS has the best performance. This means that the transformation function using CSTPS is smoother and provides more accurate registration.

![Fig. 1 Deformation Results for Points on a Line and a Semicircle.](image)

![Table 1 Comparison of Results from Fig.1(a) ($c=20$) and Fig.1(b) ($c=30$).](table)

<table>
<thead>
<tr>
<th>CSTPS</th>
<th>Cos</th>
<th>Wu</th>
<th>$\Psi_{3,3}$</th>
<th>Buhmann</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(a)</td>
<td>(b)</td>
<td>(a)</td>
<td>(a)</td>
</tr>
<tr>
<td>0.377</td>
<td>0.3441</td>
<td>0.4814</td>
<td>0.415</td>
<td>0.4398</td>
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<td>2</td>
<td>4</td>
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<td>0.120</td>
<td>0.2803</td>
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<td>3</td>
<td>3</td>
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4.2. Deformation on a Uniform Grid

To compare the image-registration accuracies of various transformation functions, a number of test images,
shown in Fig. 2(a)–2(d), were used [11]. Fig.2(a) shows an image of size 64×64 pixels containing a 4×4 uniform grid. Fig.2(b)–2(d) can be used to evaluate the performance of various transformation functions when images have linear or nonlinear geometric differences. In particular, Fig.2(c)–2(d) can be used to demonstrate the performance of the transformation functions when images have global or local nonlinear geometric differences.

![Fig. 2 (a) Uniform Grid Deformed by (b) Affine Transformation, (c) Inverse Distance Transformation, (d) Sinusoidal Transformation, (e) Sparse Landmarks, (f) Dense Landmarks.](image)

Landmarks at various densities were used to construct the transformation functions. Fig.2(e) and 2(f) show landmarks uniformly spaced at eight-pixel and four-pixel intervals respectively. Table 2 presents the Error values for the registration results obtained using different CSRBFs to map Fig.2(a) to Fig.2(b)–2(d) using the landmarks shown in Fig.2(e) and 2(f). The maximum registration error is the maximum distance between the target position of any point and its expected position. The average registration error is the mean distance between the target position of any point and its expected position. It is apparent that the Wu is best for affine registration (Fig.2(b)) with either dense or sparse landmarks. CSTPS performed best for local registration (Fig.2(c)) and for global registration with dense landmarks. The Cos performed badly with dense landmarks because it warped the grid too much and did not preserve topological relations properly. Table 2 indicates that the elastic image-registration accuracy of CSTPS is satisfactory for images with both local and global geometric differences.

<table>
<thead>
<tr>
<th>CSTPS</th>
<th>Cos</th>
<th>Wu</th>
<th>Buhmann</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRE</td>
<td>ARE</td>
<td>MRE</td>
<td>ARE</td>
</tr>
<tr>
<td>Affine</td>
<td>4.6288</td>
<td>2.67×10^4</td>
<td>1.99×10^4</td>
</tr>
<tr>
<td>Sparse(c=100)</td>
<td>0.251</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>Dense(c=100)</td>
<td>1.6412</td>
<td>0.038</td>
<td>5.06×10^9</td>
</tr>
<tr>
<td>Sparse(c=40)</td>
<td>6</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>Dense(c=100)</td>
<td>4.8654</td>
<td>4.233</td>
<td>1.36×10^16</td>
</tr>
<tr>
<td>Local</td>
<td>3.7638</td>
<td>0.344</td>
<td>5.6636</td>
</tr>
<tr>
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<td>0.5082</td>
</tr>
<tr>
<td>Dense(c=20)</td>
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<td>0.085</td>
<td>3.9471</td>
</tr>
<tr>
<td>Local</td>
<td>2</td>
<td>6</td>
<td>0</td>
</tr>
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</table>
5. Experimental Results and Discussion

5.1. Similarity to TPS

A 257×257 uniform grid was used to compare the registration results. Six and twelve random landmark pairs were generated separately to deform the 257×257 uniform grid over forty iterations. The displacement of each landmark was also randomly generated and was limited to 16 and 8 for six and twelve landmark pairs respectively. The supports of the CSRBFS were \( c = 1000 \). The difference between the deformation results after applying different CSRBFS and TPS can be computed as

\[
\text{Difference} = \log \left( \sum_{(x,y)} \left| f_{\text{TPS}}(x,y) - f(x,y) \right| + 1 \right)
\]

where \( f_{\text{TPS}}(x,y) \) is the transformation function using TPS and \( f(x,y) \) is the transformation function using a CSRBF.

Fig. 3 shows the differences obtained for six and twelve random landmark pairs respectively. Clearly, the differences between the deformation results using CSTPS and TPS are small, especially for dense landmarks (12 landmarks). Because the spacing between landmarks is small for dense cases, the support is relatively large for the small spacing between landmarks. Based on the analysis in Section 3, the larger the support, the more similar is the CSTPS to TPS.

Fig. 3 Differences Between Transformations Using CSRBFs and Using TPS.

Fig. 4 Evaluation Results for Registration of Source and Reference Images Using the CSTPS, Cos, Wu, \( \Psi_{3,1} \), and Buhmann Functions with Different Supports \( c \).
5.2. Multimodal Medical Image Registration

To evaluate the image-registration accuracies the CS RBFs, a number of medical images were used. These 256×256 pixel images with 1-mm isotropic voxel size were extracted from 3D MRI data sets. 10 slices from the T1 tomographic images were used as the source images. The set of 64 uniformly spaced source landmarks were shifted randomly with offset 4 to serve as the target landmarks. Using TPS to map the 10 slices from the T1 tomographic image using these manually defined landmark sets, the reference images were obtained.

The source images were then deformed by transformation functions using different CSRBFs with manually defined corresponding landmarks. The registration error was evaluated using the image difference measure, which is the mean of the absolute difference between the warped image and the reference image. Fig. 4 shows the evaluation results for various CSRBFs. Different supports were used to determine the influence of the degree of locality on registration accuracy. It can be seen that CSTPS used for medical image registration performed satisfactorily with both large and small supports. CSTPS was always found to provide more stable registration accuracy than the other CSRBFs.

6. Conclusions

Landmark-based transformations model deformations using a set of radial basis functions. This paper has analyzed and evaluated the performance of the CSTPS for elastic image registration. The positive definite property of CSTPS was first theoretically proved, which ensures the solvability of the equation to determine the radial basis function coefficients. Based on a simple point-matching model, it was proved that deformations using CSTPS with large supports are highly similar to those from TPS. The CSTPS has the advantage of creating a smooth deformation. For both local and global geometric differences between images, CSTPS performed well and provided satisfactory registration results.

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References


